Practical Suggestions for Mathematical Writing

Renee Bell, Borys Kadets, Padmavathi Srinivasan, Nicholas Triantafillou, and Isabel Vogt

Dedicated to our advisor, in gratitude and admiration.

The primary goal of an article is to bring the reader on a rewarding mathematical journey with the writer. Your role as a writer is analogous to that of the leader of a hike. You need to gently guide your readers through the weeds of a technical argument while making sure they fully appreciate the highlights of your work.

In this article, our goal is to share some valuable practical lessons we have learned from our advisor, Bjorn Poonen, inspired by his article [4]. We have found ourselves revisiting these ideas from time to time and would like to make them more widely known within the mathematical community. Many of the suggestions here are directly from [4], and we refer readers to the original source for these and many more wonderful concrete suggestions.

A short article like the one you are reading now cannot hope to be a comprehensive summary of strategies for writing math well. Many writers we admire have also written and spoken about other aspects of clear mathematical writing. We direct the interested reader to some of our favorite resources [2], [4], [5], [6], [7] at the end of this article.

1. Getting Ready to Write

Planning before you begin writing can save you time later. In this section, we suggest some things to do before you start writing.

1. Choose an audience

The choices you make when writing your article should depend on who you hope will read it. To ground yourself, imagine you are writing for a specific person who might read your article.[6] For example, if you are a graduate student, you may want to write for a particular fellow student in your area. You can ask yourself:

- What is your reader's background?
- What motivation will they need to put your results in context?
- How much detail would they appreciate seeing in an argument?

2. Make a detailed outline

Before writing a longer article, try to begin with a plan for the sections, main lemmas, and theorems. You can fill in the proofs later. Be aware that when you add detail to your proofs, you may discover that you need to reorganize your initial outline. You should be ready to ruthlessly revise later!

To help you make a detailed outline, explain your work to a friend. This forces you to take a step back and think about both the big picture and the highlights of your paper.

3. Choose notation well

Bad notation is the easiest way to lose your reader, and also the easiest mistake to avoid! To save your reader several hours of unnecessary confusion, it is well worth investing the time at the beginning to carefully choose externally and internally consistent notation. That is, make sure you use standard notation in the field and make sure you give related names to related objects when you have to introduce new notation.

When working on a paper with many coauthors or if you have not decided on all of your notation, be prepared to change notation later on. Changing notation with "find and replace" is time-consuming and error-prone. To save yourself unnecessary pain and to ensure that you do not overlook any instance of your previous notation, you can define a macro in LaTeX.

2. Writing a Good Introduction

The introduction has a unique and important role to play; you should think of it as a sales pitch for your article. The reality is that most readers will only have time to read your introduction. So, it should cater to both casual browsers and researchers on the lookout for a specific result. That said, a well-written introduction builds confidence that reading on will be a pleasurable experience and may entice your audience to invest the time to read the rest of the article!

Renee Bell is a Hans Rademacher Instructor at the University of Pennsylvania. Her email address is rhbell@sas.upenn.edu.

Borys Kadets is a limited term assistant professor at the University of Georgia. His email address is kadets@uga.edu.

Padmavathi Srinivasan is a limited term assistant professor at the University of Georgia. Her email address is Padmavathi.Srinivasan@uga.edu.

Nicholas Triantafillou is a postdoctoral research and teaching associate at the University of Georgia. His email address is nicholas.triantafillou@uga.edu.

Isabel Vogt is an assistant professor of mathematics at the University of Washington in Seattle. Her email address is ivogt@uw.edu.

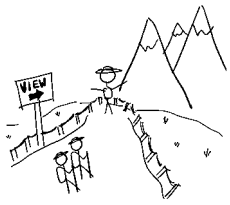
DOI: https://dx.doi.org/10.1090/noti2305

[6] *Four out of five authors of this article imagine they are writing for Isabel Vogt while working on the first draft. Five out of five authors imagine Bjorn Poonen when editing.*

When writing your introduction:

1. Get to the point
Showcase your exciting new results as soon as possible in the introduction. It is OK to postpone definitions to a later "Notation" section (and to refer readers in the introduction to this later section) if those definitions are standard enough that most readers will be able to read the introduction without them.

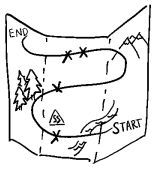
2. Make your results stand out
It is easy to miss a main result hidden in the middle of a paragraph. Use theorem environments or separate paragraphs so that busy readers can quickly scan for your main results.



3. Place your results in context
Explain how your work fits in with and diverges from previous approaches by highlighting the novel contributions while illustrating the broader landscape.

It is worthwhile to use MathSciNet® and/or Google Scholar to look through papers which cite the same foundational references as your paper. This can help you find recent related work. Be generous in citing related articles.

4. Paint a clear roadmap

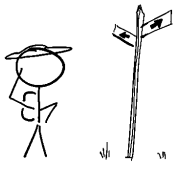


Outline the remaining sections in the paper. Highlight the main arguments in each section and how they all fit together.

This added structure gives a canvas for your reader to sketch an outline of the entire paper with the main arguments. It is valuable to have a good sense of what is to come in a lengthy technical article.

3. Building a Clear, Cohesive Narrative

3.1. Guiding text



After you have carefully planned your paper, the global structure of your argument is probably crystal clear to you. But for your reader, it might not be so clear. Some well-chosen guiding sentences can act as landmarks. They can help your reader identify where they are on the roadmap you outlined in the introduction.

1. Introduce each section
It is good practice to begin every section with a sentence indicating what will be done in that section. This serves to reorient your reader and will help them contextualize where the current section fits into the larger argument.

2. Highlight hypotheses
Make clear in your proofs where each of the hypotheses is being used. In addition, you can motivate your hypotheses by providing interesting counterexamples to your theorem when some assumptions are removed.

3. Include a running example (when possible)
Examples in a paper can

- help your readers digest new concepts,
- illustrate the main ideas of a long technical argument, and
- anchor readers who are less familiar with the topic.

Using one or two well-chosen running examples throughout the article can accomplish these goals cohesively, while saving your reader the time it would take to reorient to a new example every time. If you are unsure of what example to include, think back to when you were proving the result: there is a good chance you worked out a minimal interesting example before you discovered the full argument!

3.2. Reducing cognitive load

Most people (especially mathematicians!) can hold only a few ideas in their head at once. Big-picture ideas, technical details, and complicated equations compete for your audience's brain space as they read your article. Subdividing arguments and simplifying notation lets your reader use their mental energy to enjoy your mathematics.

1. Break up long arguments
Subdivide long proofs using lemmas. It is OK if you use a lemma only once. The goal is to minimize what your reader must keep in mind at one time.

2. Say what to remember and what to forget
If a section contains several theorems, propositions, and lemmas, but only one of them is needed in subsequent sections, mention this. (Again, this can free up memory in your reader's brain!)

3. Keep theorem statements short (within reason)
Cluttered theorems can make it difficult for your reader to figure out what you are proving. To reduce clutter, you can:

(a) Define terms before the theorem in which they are used.
(b) Isolate complicated technical hypotheses in a separate labeled environment.

That said, if your theorem requires a hypothesis which is not a standing assumption of your entire paper, you should include that hypothesis in or near your theorem statement. Do not sacrifice clarity for brevity.

4. Keep sentences simple
Sentences with a single idea are easier to read. Combine sentences (with ", and" or with a semicolon) only if it helps to clarify the logic or if the sentences are otherwise closely related.

It is OK if short sentences make your prose choppy or repetitive compared to other forms of writing. Prioritize readability and mathematical clarity over flow.

5. Start with a warm-up
If a proof naturally breaks up into parts, do the easier parts first. This is especially true if the easy and hard parts of the proof use similar ideas.

6. Anticipate what your readers will forget
If you use a definition for the first time in a while, either give a reference or include a notation section.

7. Avoid clutter
Use extra symbols only if they help your reader parse your expressions. For example:

(a) Minimize use of parentheses. For instance, $\log x$ is better than $\log(x)$. It is also traditional to omit parentheses in some expressions like $\sin 2x$. But in $\sin(x + y)$, parentheses are necessary.
(b) Usually it is not necessary to use a centered dot or other symbol for multiplication.

8. A picture is worth a thousand words
As appropriate to your discipline, consider supplementing your written argument with a schematic or diagram to clarify a complicated setup. Some tips for using figures effectively:

(a) Be careful with colors. Make your diagrams accessible to color-blind readers or in black-and-white print. For example red/green is bad. These two colors are much better. There are many palette guides online, for example [3].
(b) Write a complete argument and then supplement it with figures. It might seem natural to replace part of an argument with a picture, but you should make sure your written argument can stand alone.

3.3. Structuring proofs for maximal clarity

Determining if your own writing is clear can be challenging. When you understand your arguments and notation too well, you can navigate your proofs in large chunks, stepping through several sentences or paragraphs at a time. In contrast, your readers may comb through your proof one sentence (or less!) at a time. A detail that feels like a pebble to you can become a boulder obstructing your reader's progress. Fortunately, with some good writing and editing habits you clear the most common sources of needless confusion and frustration from your reader's path.

1. Don't leave your reader hanging
When your reader reaches the period of a sentence, they should either know why each claim up to that point is true or know that an explanation will follow. If a sentence contains a claim that will be justified later, end the sentence with ", as we now explain."

2. Clearly connect reasons and their conclusions
When a single assertion takes more than a few sentences to explain, consider turning it into a separate lemma. If that is not possible, clarify which sentences contain the justification by writing something like "Combining the previous two sentences with Lemma 5 shows that . . . ."

If a sentence contains multiple claims, make clear which explanation is the justification for which claim.

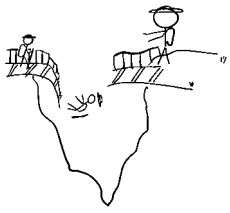
3. Declare your variables
Introduce new notation before using it in a formula so that your reader does not need to cycle back to the start of your sentence. When editing, search your document for ", where" and try to rephrase sentences whenever possible. For instance, it is better to write "Let x be a real number. Then, $x^2 + 1 \geq 0$." than to write "We have $x^2 + 1 \geq 0$, where x is any real number."

4. Avoid unnecessary "Proofs by contradiction"
Proofs by contradiction add temporary hypotheses that make it difficult to keep track of which assumptions are active. When possible, rephrase proofs by contradiction; this is especially easy when you are secretly proving the contrapositive. If proof by contradiction is unavoidable, consider turning the whole subargument into its own paragraph or lemma.

5. Explain the "obvious"
Avoid using the words "clear," "obvious," "trivial," and the like as the sole justification for a claim. It rarely takes more than a sentence to explain why something is clear. Explaining things that you consider trivial also helps you to be confident that the results are correct.

One of our other advisors, Joe Harris, likes to recount the following cautionary tale. In the final stages of proofreading a manuscript, he found a false claim, which, alarmingly, was preceded by the word "clearly." Upon searching for other instances of "clearly" and "it is not hard to see that," he found half a dozen other such errors! He concluded that he had been using the word "clearly" defensively when he did not actually have a solid argument.

3.4. Eliminating ambiguity



Mathematicians are a creative and exploratory bunch; if you leave open a path to an alternative interpretation, someone is likely to take it. By choosing a few well-placed words, you can add precision and dispel ambiguity, protecting your reader from treacherous missteps.

To spare your reader a confusing unintended detour, try the following practices. They aim to construct mathematical guardrails to direct your reader to your results in the most efficient fashion.

1. Clarify your quantifiers
Implicit quantifiers can engender a multitude of ambiguities. For example, the end of "We have $x^2 + 1 \in S$ for $x \in \mathbb{R}$" can be interpreted as "for some $x \in \mathbb{R}$" or "for all $x \in \mathbb{R}$"; this ambiguity should be resolved, even if it seems reasonable to deduce your intended meaning from context.

2. Limit usage of pronouns
Use pronouns, like "it," sparingly, and make sure you are unambiguously designating what your pronoun is referring to.

3. Beware of not-quite synonyms
In general, be careful with prepositions: the choice of "so," "for," "as," and the like can be subtle but have serious mathematical implications.

(a) For example, the words "so that" should be used to indicate purpose, not impose a condition (for which you use "such that"). Good: We include 0 in $\mathbb{N}$ so that $\mathbb{N}$ contains the size of every finite set. Bad: An abelian group is a group so that every two elements commute.
(b) Try not to use "as" as a synonym for "since," since there are many other meanings of "as," particularly in math. In [4], our advisor provides a particularly egregious example: "As x does not tend to $+\infty$, the expression e^x is bounded."
(c) Similarly, do not use "per" when what you mean is "by."

4. Avoid misplaced modifiers
The word "only" should be placed as close as possible to the word it is modifying. Consider the sentence "This set only contains the zero vector." It can be interpreted to mean either that only this set, among many, contains the zero vector, or that the only element of this set is the zero vector.

5. State what you actually prove
In your theorem statements, instead of saying two objects are isomorphic, specify a map between them and claim that it is an isomorphism. This is strictly stronger, is the form in which the theorem is more likely to be used, and is probably what you mean anyway.

6. Use a single numbering system
To avoid citation issues and make statements easier to find, use a single numbering system for all theorems, lemmas, etc. Using the amsart document class, put

```
\newtheorem{theorem}{Theorem}[section]
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{proposition}[theorem]{Proposition}
```

in your preamble.

7. Cite precisely
When citing other work, include a precise reference to a theorem number or page number (unless you really mean to cite the entire work!). This saves your reader the time and confusion of wading through an entire reference.

For an arXiv preprint, include the version number or precise date. For preprints elsewhere on the web, give the URL and date of the manuscript or date downloaded. Similarly, cite version numbers of software or computer code. This way, even if a new version is uploaded, your reader can figure out what you were referring to.

It is worth noting that looking for these ambiguities requires some care. They may not always read or sound like ambiguities in a research talk or conversation at the blackboard, where writing is supplemented by subtle but essential verbal cues. Even cadence and tone of voice can indicate meaning, but these are all lost on paper; you should ensure your reader will stay on the path when you are not there to guide them. It is natural to overlook such ambiguities, but another person can easily find them for you, so solicit feedback from a friend or mentor!

3.5. Math and grammar

The same general rules of English grammar hold for mathematical writing. In this section, we focus on some specific grammatical suggestions for clearly integrating technical mathematics into your writing.

1. Structure your sentences so that they are easy to parse. The main verb of your sentence should be written out in words, not in symbols. Avoid splitting the subject of a sentence or clause into some words outside a formula, together with a portion of the formula. A particularly flagrant example illustrating both of these issues is: "For an elliptic curve $y^2 = x^3+ax+b$, the discriminant $\Delta = -16(4a^3 + 27b^2)$."
   Here the second time the symbol "=" appears, it functions as the verb of the sentence, and the subject "the discriminant Δ" is split between the formula and the rest of the sentence. If one were to parse the first part of the sentence in the manner of the second, y^2 would be an elliptic curve! An easy fix would be to replace the second "=" with the word "equals."
2. Use the same punctuation in a sentence with a displayed equation as in a sentence with an in-line equation. In particular, the line before a displayed equation often does not need to end in a colon. Also remember that a displayed equation at the end of a sentence should end with a period.
3. Do not use abbreviations like WLOG, iff, and s.t. in your writing. Similarly, write out the English words for logical symbols (like "for all" instead of $\forall$) unless you are using them in equations in formal logic.
4. Do not start a sentence with a symbol. (Your reader may miss that you have begun a new sentence!)
5. Capitalize the word "theorem" (or "definition" or "section") only when you are referring to a theorem by number.

There are a few special cases where mathematical writing does not follow the standard rules you may have learned in an English class.

1. Use numerals for numbers in math papers. A small counting number can be spelled out if it is not representing a mathematical quantity. For example, write: "consider two genus 2 curves...."
2. The prefix "non" before a common math word does not usually need a hyphen. For example, write nonempty, nontrivial, and nonzero.

4. Community Practices

Finally, it is important to remember that our writing enters and influences the canon making up mathematics. As such, each of us has a role to play in fostering an open and welcoming community of researchers.

1. Announced results which take a very long time to appear (or never appear!) are detrimental to progress in your field. For this reason, cite "forthcoming work" only if a publicly available preprint exists. (This is consistent with the AMS Ethical Guidelines [1, Section I].)
2. At best, it is an unpleasant experience to not be able to follow a step of a proof that the author claims is trivial. At worst, such unintentional "microslights" can pile up and deter those who are prone to doubt themselves [7]. Consider using phrases such as "a standard calculation involving ... shows..." to convey the degree of novelty while pointing your reader in the right direction.

5. Closing Thoughts

Writing and revising can be a lengthy and laborious process. It is normal to feel stuck. To gain fresh perspective, enlist a second pair of eyes or simply step away for a while.

Our advisor likes to recount a possibly apocryphal story about how Jean-Pierre Serre would lock up old drafts in a drawer and write several new drafts from scratch until his drafts converged!

Writing well requires a significant investment on the part of the writer to achieve the golden ratio of clarity and length. That said, communicating your excitement to your reader is ample reward for this effort. We hope this article gives you several practical suggestions that you can use for your own writing process to achieve this worthy goal. Happy writing!

References

[1] Ethical guidelines of the American Mathematical Society. Available at http://www.ams.org/sec-ethics. Retrieved on February 12, 2021.

[2] Donald Knuth, *Course notes on mathematical writing*. Available at https://jmlr.csail.mit.edu/reviewing-papers/knuth_mathematical_writing.pdf. Retrieved on February 12, 2021.

[3] David Nichols, *Coloring for colorblindness*. Available at https://davidmathlogic.com/colorblind/. Retrieved on February 12, 2021.

[4] Bjorn Poonen, *Practical suggestions for mathematical writing*. Available at http://www-math.mit.edu/~poonen/papers/writing.pdf. Retrieved on February 12, 2021.

[5] Jean-Pierre Serre, *How to write mathematics badly*. Available at https://www.youtube.com/watch?v=ECQyFzzBHlo. Retrieved on February 12, 2021.

[6] Francis Su, *Some guidelines for good mathematical writing*, MAA Focus **35** (2015), no. 4, 20–22. Available at https://scholarship.claremont.edu/hmc_fac_pub/1145. Retrieved on February 12, 2021.

[7] Francis Su, *Mathematical microaggressions*, MAA Focus **35** (2015), no. 5, 36–37. Available at http://ee.usc.edu/stochastic-nets/teaching-resources/documents/Su-Mathematical-Microaggressions.pdf. Retrieved on February 12, 2021.

The academic family celebrating Isabel's and Nicholas' thesis defenses in April 2019. (From left to right: Renee Bell, Bjorn Poonen, Isabel Vogt, Padma Srinivasan, Borys Kadets, Nicholas Triantafillou, Vishal Arul.)

Credits

Illustrations and author photo are courtesy of Isabel Vogt.